Math 1.1

I Semester M.Sc. Mathematics Examination, May 2011
ALGEBRA

Time : 3 Hours Max. Marks : 80

Note : *1) Answer **any five** questions.*
*2) **All** questions carry **equal** marks.*

1. a) State and prove Lagrange theorem for finite groups.

   b) Let $f : G \rightarrow G'$ be a group homomorphism. Then prove that ker f is a normal subgroup of G. Moreover prove that f is a one-one mapping if and only if ker f = {e}. **(8+8)**

2. a) Prove that every permutation $\sigma \in S_n$ can be expressed as a product of disjoint cycles.

   b) Prove the class equation of the group G. **(8+8)**

3. a) State and prove the first Sylow theorem.

   b) Show that any group of order $5^2.7^2$ is abelian. **(8+8)**

4. a) Show that any integral domain can be embedded in a field.

   b) Let R be a commutative ring with identity. Then prove that R is a field if and only if the only ideals of R are {0} and R itself. **(10+6)**

5. a) State and prove the fundamental theorem of homomorphism for rings.

   b) Let R be a commutative ring with identity. Prove that an ideal P in R is a prime ideal if and only if R/P is an integral domain. **(8+8)**

6. a) Prove that in a Unique factorization domain, an element is a prime if and only if it is irreducible.

   b) Let F be a field and $f(x) \in F[x]$. Then prove that $\alpha \in F$ is a root of f(x) if and only if $(x - \alpha)$ divides f(x). **(8+8)**

7. a) Let W be a subspace of a finite-dimensional vector space V. Then prove that W is finite - dimensional and dim W $\leq$ dim V.

   b) If $F \subseteq K \subseteq L$ are fields, then prove that [L : F] = [L : K] [K : F]. **(8+8)**

8. a) Prove that any splitting field of a polynomial over F is a normal extension of F.

   b) State and prove the Primitive Element Theorem. **(8+8)**
