First Year M.Sc., Degree Examinations

September / October 2015

(Directorate of Distance Education)

MATHEMATICS

Paper – PM 10-01: ALGEBRA

Time: 3hrs.] *[Max. Marks: 70/80*

Instructions to candidates:

i) ***Students who have attended 30 marks IA scheme will have to answer for total 70 Marks.***

ii) ***Students who have attended 20 marks IA scheme will have to answer for total 80 Marks.***

iii) ***Answer any FIVE questions for both 70-80 marks scheme and Question No. (9) in Section – B is compulsory for 80 marks.***

SECTION – A

1. a) Prove that every permutation can be expressed as a product of transpositions in infinitely many ways.

   b) Prove that the set $I(G)$ of all inner automorphisms of a group G is a normal subgroup of $A(G)$, the group of all automorphisms of G and that $I(G)$ is isomorphic to G/Z, where Z is the centre of G.

   c) State and prove Cauchy's theorem for finite groups. (4 + 5 + 5)

2. a) If P is the only Sylow p-subgroup of G, then prove that P is normal in G and conversely.

   b) If G is a group of order 231, show that 11-Sylow subgroup of G is contained in the centre of G.

   c) Define a Solvable group. If G is a solvable group and H is a subgroup of G, then show that H is solvable. (4 + 6 + 4)

3. a) Let R be a commutative ring with unit element whose only ideals are (O) and R itself. Prove that R is a field.

   b) Prove that the ideal S of the ring I of all integers is maximal if and only if S is generated by some prime integer.

   c) Define maximal ideal of a ring R. If R^1 is a ring of all real valued continuous functions defined on the closed interval $[0,1]$ and if $M = \{f(x) \in R : f(1/3) = 0\}$, show that M is a maximal ideal of R. (4 + 4 + 6)

4. a) If R is a principal ideal domain, prove that every non-zero prime ideal of R is maximal.

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b) State and prove unique factorization theorem for Euclidean rings.

c) State and prove Eisenstein criterion for irreducibility of a polynomial over a rational field. (4 + 6 + 4)

5. a) Prove that $L(S)$ is a subspace of a vector space V, where $L(S)$ is the linear span of a subset S of V.

b) If V and W are finite dimensional vector spaces with dimensions m and n respectively, then show that Hom (V, W) is also a vector space with dimension 'mn'.

c) If W is a subspace of a finite dimensional vector space V, define the annihilator $A(W)$ of a subspace W. Further show that

i) $A(W_1+W_2)=A(W_1)\cap A(W_2)$

ii) $A(W_1\cap W_2)=A(W_1)+A(W_2)$ (4 + 5 + 5)

6. a) Define rank and nullity of a linear transformation. If V is a finite dimensional vector space and if $T\in A(V)$, then prove that dim $V=\text{rank}T+\text{nullity}T$.

b) If S, $T\in A(V)$ and if S is regular then show that T and STS^{-1} have the same minimal polynomial.

c) Let V be a vector space of dimension 'n' over a field F and let $T\in A(V)$. If $m_1(T)$ and $m_2(T)$ are the matrices of T relative to the bases $\{v_1, v_2, \ldots, v_n\}$ and $\{w_1, w_2, \ldots, w_n\}$ respectively, show that there is an invertible matrix C in F_n such that $m_2(T)=C\,m_1(T)C^{-1}$. (4 + 4 + 6)

7. a) If $T\in A(V)$ is nilpotent, of index of nilpotence n_1, then prove that there exists a basis of V such that the matrix of T in this has the form $\begin{pmatrix} M_{n_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & M_{n_r} \end{pmatrix}$, where $n_1\ge n_2\ge\ldots\ge n_r$ and where $n_1+n_2+\ldots+n_e=\dim V(F)$.

b) Show that the matrix $A=\begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 1 & 0 \end{pmatrix}$ is not similar to $B=\begin{pmatrix} 1 & 1 & 1 \\ -1 & -1 & -1 \\ 1 & 0 & 0 \end{pmatrix}$.

c) Determine all possible Jordon canonical forms for a linear operator T on V whose characteristic polynomial $\Delta(t)=(t-2)^3\,(t-t)^2$. In each case, find the minimal polynomial $m(t)$. (6 + 4 + 4)

8. a) Prove that every finite extension K of a field F is algebraic.

b) If a and b in K are algebraic over F of degrees m and n respectively, then prove that $a\pm b$, $a.b$ and a/b $(b\neq 0)$ are algebraic over F of degrees at most 'mn'.

c) Prove that a polynomial of degree *n* over a field can have atmost '*n*' roots in any extension field. (5 + 4 + 5)

SECTION – B

9\. a) Let V be the vector space of all polynomials p from R into R which have degree 2 or less. Define three linear functionals on V by

$$f_1(p)=\int_0^1 p(x)dx;\ \ f_2(p)=\int_0^2 p(x)dx;\ \ f_3(p)=\int_0^{-1} p(x)dx;$$ Show that $\{f_1,\ f_2,\ f_3\}$ is a basis of $\hat{V}$. Determine a basis of V such that $\{f_1,\ f_2,\ f_3\}$ is its dual basis.

b) Let V be the vector space of all 2×2 matrices over R and let $M=\begin{pmatrix}1 & 2\\ 3 & 4\end{pmatrix}$. If T is a linear operator on V defined by $T(A)=MA$, find the trace of T. (6 + 4)

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First Year M.Sc., Degree Examinations

September / October 2015

(Directorate of Distance Education)

MATHEMATICS

Paper – PM 10-02: DPA-520: ANALYSIS – I

Time: 3hrs.] *[Max. Marks: 70/80*

Instructions to candidates:

i) ***Students who have attended 30 marks IA scheme will have to answer for total 70 Marks.***

ii) ***Students who have attended 20 marks IA scheme will have to answer for total 80 Marks.***

iii) ***Answer any FIVE questions for both 70-80 marks scheme and Question No. (9) in Section – B is compulsory for 80 marks.***

SECTION – A

1. a) If x, y are real numbers, $x > 0$, show that there exists a positive integer *n,* such that $nx > y$.

   b) For any $x > 0$, and for every positive integer *n*, show that there exist a unique $y \ni x = y^n$.

   c) Prove that between any two real numbers there exists infinitely many rationals.

   (5 + 5 + 4)

2. a) Define a countable set. Prove that countable union of a countable sets is countable.

   b) Prove that $(0,1)$ is uncountable.

   c) Define a compact set and connected set and give examples for each. (6 + 4 + 4)

3. a) Prove that a set is compact if and only if it is closed and bounded.

   b) Show that every infinite subset of $\mathbb{R}^k$ has a limit point in $\mathbb{R}^k$.

   c) With example, verify that R' is compact or not. (5 + 5 + 4)

4. a) Prove that a monotonic sequence $\{x_n\}$ converges if and only if it is bounded.

   b) Show that $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$ and further prove that e is an irrational number.

   c) If $p > 0$, and α is real, then show that $\lim_{n\to\infty}\frac{n^\alpha}{(1+p)^n} = 0$. (5 + 5 + 4)

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5. a) Define a continuous function. Show that continuous image of a compact set is compact.

   b) Let X, Y be metric spaces, X is compact and $f : X \to Y$ continuous on X. Prove that f is uniformly continuous on X.

   c) If $f(x) = |x|^3$, compute $f'(x)$, $f''(x)$ for all real x, and show that $f^{(3)}(0)$ does not exists. (5 + 5 + 4)

6. a) State and prove Cauchy's mean value theorem.

   b) State and prove Lagrange mean value theorem.

   c) Let $f : (a,b) \to R'$. If f' exists and non-negative, then prove that $f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}(f(a) + f(b))$. (5 + 5 + 4)

7. a) Prove that $f \in \mathbb{R}(\alpha)$ on $[a, b]$ if and only if for every $\varepsilon > 0$ there exists a partition P of $[a, b]$ such that $U(P, f, \alpha) - L(P, f, \alpha) < \varepsilon$.

   b) If f is continuous on $[a, b]$, then prove that $f \in \mathbb{R}(\alpha)$ on $[a, b]$.

   c) If $|f| \in R(\alpha)$ on $[a, b]$ does $f \in R(\alpha)$ on $[a, b]$? Justify. (5 + 5 + 4)

8. a) State and prove the Fundamental theorem of Integral Calculus.

   b) Define a function of bounded variation. If $f \in BV[a, b]$, does $\sqrt{f} \in BV[a, b]$? (8 + 6)

SECTION – B

9. a) Are $\mathbb{Z}$, $\mathbb{Q}$ compact? Justify.

   b) Examine the function

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$$

   for Riemann integrability on $[0,1]$. (5 + 5)

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First Year M.Sc., Degree Examinations

September / October 2015

(Directorate of Distance Education)

MATHEMATICS

Paper – PM 10-03: DPA 530: ANALYSIS – II

Time: 3hrs.] *[Max. Marks: 70/80*

Instructions to candidates:

i) ***Students who have attended 30 marks IA scheme will have to answer for total 70 Marks.***

ii) ***Students who have attended 20 marks IA scheme will have to answer for total 80 Marks.***

iii) ***Answer any FIVE questions for both 70-80 marks scheme and Question No. (9) in Section – B is compulsory for 80 marks.***

SECTION – A

1. a) State and prove Marten's theorem.

   b) If $\sum a_n$ is a series of complex numbers which converges absolutely, then show that every rearrangement of $\sum a_n$ converges and all converge to the same sum.

   c) Give an example to show that product of two convergent series need not be convergent. (5 + 5 + 4)

2. a) State and prove Cauchy's criterion for uniform convergence.

   b) Prove the necessary and sufficient condition for uniform convergence.

   c) Prove that the series $\sum_{n=0}^{\infty} \frac{x^n}{n^2}$ converges uniformly on $[-1, 1]$. (5 + 4 + 5)

3. a) Prove the theorem on the relation between uniform convergence and continuity.

   a) Show that there exists a real continuous function on a real line which is nowhere differentiable.

   b) Show that $\sum (-1)^n \frac{x^2+n}{n^2}$ converges uniformly in every bounded interval, but does not converge absolutely for any values of x. (5 + 5 + 4)

4. a) Let α be monotonically increasing on $[a, b]$. Suppose $f_n \in R(\alpha)$ on $[a, b]$ and if $f(x) = \sum_{n=1}^{\infty} f_n(x)$; $(a \le x \le b)$, the series converge uniformly on $[a, b]$. Then show that $\int_a^b f\, d\alpha = \sum_{n=1}^{\infty} \int_a^b f_n\, d\alpha$.

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b) If $f_n(x)$ is a sequence of continuous functions on E and if $f_n(x) \to f(x)$ uniformly on E. Then show that $f(x)$ is continuous on E.

c) Show that the sequence $\{f_n\}$, where $f_n(x) = \frac{x}{1+nx^2}$ converges uniformly to a function $f(x)$ on $[0, 1]$ and that the function $f'(x) = \lim_{n\to\infty} fn^1(x)$. (5 + 5 + 4)

5. a) If f and g are two positive functions on $[a, b]$ such that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = l$, where l is a non zero finite number, then prove that two integrals $\int_a^b f(x)\, dx$ and $\int_a^b g(x)\, dx$ converge and diverge together at $'a'$.

b) Derive exponential function. Show that

(i) e^x is continuous and differentiable for all x.

(ii) e^x is strictly increasing function of x and $e^x > 0$

(iii) $e^{x+y} = e^x\, e^y$

c) Test the convergence of the following integrals:

$$(i) \int_1^2 \frac{xdx}{\sqrt{x-1}} \quad (ii) \int_0^1 \frac{dx}{\sin x}.$$ (5 + 5 + 4)

6. a) If f is positive function on $(0, \infty)$ such that $(i)\ f(x+1) = x\, f(x)$ $(ii)\ f(1) = 1$ (iii) $\log f$ is convex, then show that $f(x) = \Gamma(x)$.

b) Define Beta and Gamma functions. If $x > 0$ and $y > 0$ then show that

$$\beta(x, y) = \frac{\Gamma_x\, \Gamma_y}{\Gamma_{x+y}} = \int_0^1 t^{\,x-1}\, (1-t)^{y-1}\, dt.$$

c) Prove that $\Gamma_x = \frac{2^{x-1}}{\sqrt{\pi}} \Gamma\left(\frac{x}{2}\right) \Gamma\left(\frac{x+1}{2}\right)$. (4 + 5 + 5)

7. a) Prove that to every $A \in L(\mathbb{R}^n, \mathbb{R}^1)$ corresponds a unique $y \in \mathbb{R}^n$ such that $Ax = x.y$ and also $\|A\| = y$.

b) Prove that if f maps an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^m$, then $f \in C'(E)$ iff the partial derivatives $D_j f_i$ exists and are continuous on E for $1 \le i \le m$ and $1 \le j \le n$.

c) Let $f(x,\ y)=\begin{cases}\dfrac{xy}{x^2+y^2} & if \quad (x,y)\neq(0,0)\\ 0 & if \quad (x,y)=(0,0)\end{cases}$.

Prove that $(D_1 f)(x,\ y)$ and $(D_2 f)(x,\ y)$ exists at every point of $\mathbb{R}^2$ but f is not continuous at $(0,\ 0)$. (5 + 5 + 4)

8\. a) State and prove implicit function theorem.

b) If $f:\mathbb{R}^2 \text{x}\ \mathbb{R}^3 \to \mathbb{R}^2$ is defined by $f=(f_1,\ f_2)$ and

$f_1(x_1,\ x_2,\ y_1,\ y_2,\ y_3)=2e^{x_1}+x_2y_1-4y_2+3$, $f_2(x_1,\ x_2,\ y_1,\ y_2,\ y_3)=x_1-6x_2+2y_1-y_3$

and if $a=(0,\ 1)$ and $b=(3,\ 2,\ 7)$ then $f(a,\ b)=0$. Show by implicit function theorem, there exists a mapping g such that $g(3,\ 2,\ 7)=(0,\ 1)$. Further compare $[g^1\ (3,\ 2,\ 7)]$. (8 + 6)

SECTION – B

9\. a) State and prove Taylor's theorem.

b) (i) If $x+y+z=u,\ y+z=uv$ and $z=uvw$ then show that $\dfrac{\partial(x,\ y,\ z)}{\partial(u,\ v,\ w)}=u^2v$

(ii) Show that $2x^4-3x^2y+y^2$ has neither maximum nor a minimum at $(0,\ 0)$. (5 + 5)

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First Year M.Sc., Degree Examinations

September / October 2015

(Directorate of Distance Education)

MATHEMATICS

DPA 540: Paper: PM 10.04: DIFFERENTIAL EQUATIONS

Time: 3hrs.] *[Max. Marks: 70/80*

Instructions to candidates:

i) ***Students who have attended 30 marks IA scheme will have to answer for total 70 Marks.***

ii) ***Students who have attended 20 marks IA scheme will have to answer for total 80 Marks.***

iii) ***Answer any FIVE questions for both 70-80 marks scheme and Question No. (9) in Section – B is compulsory for 80 marks scheme.***

SECTION – A

1. a) State and prove existence and uniqueness theorem for an I.V.P of the form $y'' + a_1 y' + a_2 y = 0,\ y(x_0) = \alpha,\ y'(x_0) = \beta$

   b) Find the solution of i) $y'' + y = \sec x \tan x$ *ii)* $y'' - y' + 2y = e^{-x}$. (8 + 6)

2. a) Show that the eigen values of a B.V.P $(px^1)^1 + qx + \lambda rx = 0$ on $[a, b]$ are reals provided $r(t) \neq 0$ on $[a, b]$.

   b) Obtain solution for $(1 - x^2)\, y'' - 2xy' + \alpha(\alpha + 1)y = 0$ about $x = 0$ in the form of power series. (7 + 7)

3. a) Show that $0, 1$ and ∞ are regular singular points of hyper geometric equation $x(1-x)y'' + [c - (a + b + 1)x]y' - aby = 0$

   b) Prove the following :

   i) $(2n+1)x\, p_n = (n+1)P_{n+1} + n\, P_{n-1}$

   ii) $P_n = \dfrac{1}{2^n n!} \dfrac{d^n}{dx^n}(x^2 - 1)^n$. (6 + 8)

4. a) Show that $(1 - 2xt + t^2)^{-1/2} = \sum_{n=0}^{\infty} t^n\, P_n(x)$.

   b) Find the solution of Bessel's equation $xy'' + y' + xy = 0$ about $x = 0$ using Frobenius method. (7 + 7)

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5. a) State and prove orthogonal property of Bessel's Equation.

   b) Using Picards method, obtain the solution for $y' = -2xy^2$, $y(0) = 1$, choose $|x| \le 1/2$, $|y| \le 1/2$. (8 + 6)

6. a) Find the solution for $y'' - 7y' + 6y = \sin x$, $y(0) = 0$, $y'(0) = 1$ using Laplace transform method.

   b) Find the Laplace transform of $i)\ \dfrac{1}{\sqrt{t}}$ $ii)\ \sin\sqrt{t}$.

   c) Find inverse Laplace transform of $\log\left(s^2 + 1\right)$. (7 + 4 + 3)

7. a) Find the complete integral surface of a linear partial differential equation $(2xy - 1)\,p + \left(z - 2x^2\right)q = 2(x - yz)$, given $z = 0$, $x + y = 0$.

   b) Obtain complete integral of $\left(p^2 + q^2\right)y = qz$ which passes through $y = 0$, $z^2 = 4x$ using charpit's method. (7 + 7)

8. a) Using variable separable method, solve a partial differential equation $u_t = c^2 u_{xx}$, $0 \le x \le l$, $t \ge 0$ subjected to the boundary conditions $u(0, t) = K_1$, $u(l, t) = K_2$ and an initial condition $u(x, 0) = \phi(x)$.

   b) Classify and reduce the following equation into its canonical form, given $u_{xx} = x^2 u_{yy}$. (7 + 7)

SECTION – B

9. a) State and prove Picards theorem for existence and uniqueness solution of first order differential equation $y' = f(x, y)$ satisfying the initial condition $y(x_0) = y_0$. (10)

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