(DM 21)

M.Sc.(Final) DEGREE EXAMINATION, DECEMBER – 2015

(Second Year)

MATHEMATICS

Paper – I : Topology and Functional Analysis

Time : 3 Hours **Maximum Marks: 80**

Answer Any five questions selecting at least two from each section.

All questions carry equal marks.

SECTION-A

1) a) Let X be a topological space and A a subject of X. Then show that

i) $\bar{A} = A \cup D(A)$; and

ii) A is closed $\Leftrightarrow A \supseteq D(A)$.

b) Let X be a second countable space . Then show that any open base for X has a countable sub class which is also an open base.

2) a) Prove that a topological space is compact if every basic open cover has a finite sub cover.

b) State and prove the Heine-Borel theorem.

3) a) State and prove Tychonoff's theorem.

b) Prove that every sequentially compact metric space is compact.

4) a) Prove that the product of any non-empty class of Hausdorff space is a Hausdorff space.

b) Prove that the range of a continuous real function defined on a connected space is an interval.

5) State and prove the urysohx imbedding theorem.

SECTION-B

6) Let N and N' be normed linear spaces and T is a linear transformation of N into N'. Then show that the following on T are equivalent to one another.

a) T is continuous.

b) T is continuous at the origin, in the sense that $x_n \to 0 \Rightarrow T(x_n) \to 0$;

c) There exists a real number $K \geq 0$ with rhe property that $\|T(x)\| \leq K\|x\|$ for every $x \in N$;

d) If $S = \ x : \|x\| \leq 1 \ $ is the closed unit sphere in N, then its image T(S) is a bounded set in N'.

7) a) State and prove the open mapping theorem.

b) State and prove the uniform boundedness theorem.

8) a) State and prove Bessel's inequality.

b) If M is a proper closed linear subspace of a Hilbert space H, then show that there exists a non-zero vector z_0 in H such that $z_0 \perp M$.

9) a) If T is an operator on H for which $(Tx,x)=0$ for all x, then prove that T=0.

b) If N_1 and N_2 are normal operators on H with the property that either commutes with the adjoint of the other, then show that $N_1 + N_2$ and $N_1 N_2$ are normal.

10) a) If P is the projection on a closed linear sulaspace M of H, then show that M is invariant under an operator $T \Leftrightarrow TP=PTP$.

b) If P and Q are the projections on closed linear subspaces M and N of H, then show that $M \perp N \Leftrightarrow PQ=0 \Leftrightarrow QP=0$.

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M.Sc.(Final) DEGREE EXAMINATION, DECEMBER – 2015

(Second Year)

MATHEMATICS

Paper - II: Measure and Integration

Time : 3 Hours **Maximum Marks: 80**

Answer Any five questions

All questions carry equal marks

1) a) State the axiom of Archimedes. Show that between any two real numbers x and y there is a rational number r such that $x< r <y$.

b) Prove that the set of all finite sequences from a countable set is also countable.

2) Prove that the outer measure of an interval is its length.

3) a) Prove that the interval (a,∞) is measurable.

b) Let $\mathrm{E} \subset [0,1)$ be a measurable set. Then show that for each $y \in [0,1)$ the set $\mathrm{E} \dotplus y$ is measurable and $\mathrm{m}(\mathrm{E} \dotplus y)=\mathrm{mE}$.

4) a) Let f be a bounded function defined on $[a,b]$. If f is Riemann integrable on $[a,b]$, then show that it is measurable and $\mathrm{R}\int_a^b f(x)dx = \int_a^b f(x)dx$.

b) State and prove Bounded convergence theorem.

5) a) If $\{f_n\}$ is a sequence of non negative measurable functions and $f_n(x) \to f(x)$ a-e on a set E, then show that $\int_{\mathrm{E}} f \leq \mathrm{Lim} \int_{\mathrm{E}} f_n$.

b) State and prove monotone convergence theorem.

6) Let f be an increasing real –valued function on the interval$[a,b]$. Then show that f is differentiable almost everywhere. The derivative f' is measurable, and $\int_a^b f'(x)dx \le f(b) - f(a)$.

7) a) State and prove Hölder inequality.

b) Prove that a normed linear space X is complete if and only if every absolutely summable series is summable.

8) a) State and prove Hahn decomposition theorem.

b) Prove that every measurable subset of a positive set is itself positive. The union of a countable collection of positive sets is positive.

9) a) Suppose that to each α in a dense set D of real numbers there is assigned a set $B_\alpha \in B$ such that $B_\alpha \subset B_\beta$ for $\alpha < \beta$. Then show that there is a unique measurable extended real-valued function f on X such that $f \le \alpha$ on B_α and $f \ge \alpha$ on $X \sim B_\alpha$.

b) State and prove Lebesgue decomposition theorem.

10) a) Prove that the set function μ^* is an outer measure.

b) If μ^* is a Caratheodary outer measure with respect to Γ, then show that every function in Γ is μ^* - measurable.

M.Sc.(Final) DEGREE EXAMINATION, DECEMBER – 2015

Second Year

MATHEMATICS

Paper - III: Analytical Number Theory and Graph Theory

Time : 3 Hours **Maximum Marks: 80**

Answer Any five questions

Selecting atleast two from each section

All questions carry equal marks

SECTION-A

1) a) If F has a continuous derivative f' on the interval $[y,x]$, where $0<y<x$, then prove that

$$\sum_{y<n\le x} f(n) = \int_y^x f(t)\,\mathrm{dt} + \int_y^x t-[t]\;\; f'(t)dt + f(x)\;\; [x]-x\;\; - f(y)\;\; [y]-y\;\; .$$

b) Prove that if $x \ge 1$ we have $\sum_{n\le x} \frac{1}{n} = \log x + c + 0\left(\frac{1}{x}\right)$.

2) a) Prove that for $x \ge 1$ we have $\sum_{n\le x} \mu(n)\left[\frac{x}{n}\right] = 1$ and $\sum_{n\le x} \wedge(n)\left[\frac{x}{n}\right] = \log[x]!$

b) Prove that for all $x \ge 1$ we have $\left|\sum_{n\le x} \frac{\mu(n)}{n}\right| \le 1$, with equality holding only if x<2.

3) a) Prove that for x>0, we have, $0 \le \frac{\psi(x)}{x} - \frac{\nu(x)}{x} \le \frac{(\log x)^2}{2\sqrt{x}\log 2}$.

b) State and prove Selberg's asymptotic formula.

4) a) Prove that for $n \ge 1$, the n^{th} prime p_n satisfies the inequalities $\frac{1}{6} n\log n < p_n < 12\left(n\log n + n\log\frac{12}{e}\right)$.

b) Let F be a real or complex valued function defined on $(0,\infty)$ and let $G(x)=\log x\sum_{n\le x}F\left(\frac{x}{n}\right)$ then prove that $F(x)\log x+\sum_{n\le x}F\left(\frac{x}{n}\right)\wedge(n)=\sum_{d\le x}\mu(d)G\left(\frac{x}{d}\right)$.

5) a) Show that in a connected graph G with exactly 2K odd vertices, there exist K edge-disjoint sub graphs such that they together contain all edges of G and that each is a unicursal graph.

b) Show that if a graph has exactly two vertices of odd degree, there must be a path joining these two vertices.

SECTION-B

6) a) Prove that a connected graph G is an Euler graph if and only if it can be decomposed into circuits.

b) Prove that in a complete graph with n vertices there are $\frac{(n-1)}{2}$ edge-disjoint Hamiltonian circuits, if n is an odd number ≥3.

7) a) Prove that a graph is a tree of and only if it is minimally connected.

b) Show that Every connected graph has at least one spanning tree.

8) a) Prove that Every circuit has an even number of edges in common with any cut-set.

b) Prove that if G_1 and G_2 are two 1-isomorphic graphs, the rank of G_1 equals the rank of G_2 and the nullity of G_1 equals the nullity of G_2.

9) a) Prove that a connected planar graph with n vertices and e edges has e-n+2 regions.

b) Prove that a necessary and sufficient condition for two planar graphs G_1 and G_2 to be duals of each other is as follows: There is a one to-one correspondence between the edges in G_1 and the edges in G_2 such that a set of edges in G_1 forms a circuit if and only if the corresponding set in G_2 forms a cut-set.

10) a) Prove that the ring sum of two circuits in a graph G is either a circuit or an edge-disjoint union of circuits.

b) Prove that the set of circuit vectors corresponding to the set of fundamental circuits, with respect to any spanning tree, forms a basis for the circuit subspace W_r.

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M.Sc.(Final) DEGREE EXAMINATION, DECEMBER – 2015

(Final Year)

Mathematics

Paper – IV : RINGS AND MODULES

Time : 3 Hours **Maximum Marks: 80**

Answer Any five questions.

All questions carry equal marks.

1) a) Show that in any ring the following identities hold :

$a0 = 0 = 0a$, $(-a)(-b) = ab$.

b) If ϕ is a homomorphism of a ring R into another ring, then prove that $\phi R \cong R|\phi^{-1}(0)$, Where ϕR is called the image, $\phi^{-1}0 = \{r \in R \mid \phi / r = 0\}$, *ker* ϕ.

2) Prove that the following statements are equivalent :

a) R is isomorphic to a finite direct product of rings R_i(i=1, 2, -----, n).

b) There exist central orthogonal idempotents $e_i \in R$ such that $1 = \sum_{i=1}^{n} e_i$ and $e_i R \cong R_i$.

c) R is a finite direct sum of ideals $k_i \cong R_i$.

3) a) Let C be a submodule of A_R. Prove that every submodule of A/C has the form B/C where C CBCA.

b) Prove that a finite direct product of modules is Artinian if and only if each factor is Artinian.

4) a) Prove that every maximal ideal in a commutative ring is prime.

b) If r is nilpotent. Show that 1-r is a unit.

5) a) If R is a commutative ring then prove that Q(R) is regular if R is semiprime.

b) If R is a Boolean ring, then prove that Q(R) is a Boolean ring.

6) a) Prove that the prime radical of R is the set of all strongly Nilpotent elements.

b) Prove that the radical is an ideal and R/Rad R is semiprimitive.

7) a) If B is a submodule of A and C is maximal among the submodules of A such that $B \cap C = 0$, then prove that B+C is large.

b) Prove that Rad A=0 if L(A) is complemented.

8) a) Prove that in a right Artinian ring, the radical is the largest nilpotent ideal.

b) Prove that every finitely generated right ideal is principal in a regular ring.

9) a) Prove that every free module is projective.

b) Prove that every module is isomorphic to a factor of free module.

10) a) Prove that an abelian group is injective if and only if it is divisible.

b) Prove that there is a canonical monomorphisim of M into $(M^*)^*$.

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