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Total No. of Questions : 10] **[Total No. of Pages : 02**

M.Sc. DEGREE EXAMINATION, MAY – 2017

SECOND YEAR

MATHEMATICS

Topology and Functional Analysis

Time : 3 Hours **Maximum Marks: 70**

Answer any five questions

Selecting atleast two from each section

All questions carry equal marks

SECTION - A

Q1) a) Show that a subset of a topological space is closed if and only if it contains its boundary.

b) Let f be a one-to-one mapping of one topological space onto another. Then show that f is a homomorphism if and only if both f and f^{-1} are continuous.

Q2) a) Prove that any closed subspace of a compact space is compact.

b) Prove that any continuous mapping of a compact metric space into a metric space is uniformly continuous.

Q3) a) Prove that every closed and bounded subspace of the real line is compact.

b) Show that a compact metric space is separable.

Q4) a) Show that every compact subspace of a Hausdorff space is closed. Deduce that a one-to-one continuous mapping of a compact space onto a Hausdorff space is a homomorphism.

b) Show that a closed subspace of a normal space is normal.

Q5) a) State and prove the Urysohn's Lemma.

b) Show that a topological space is connected if and only if every non empty proper subset has a non empty boundary.

SECTION - B

Q6) a) Let N be a non zero normed linear space. Prove that N is a Banach space if and only if $\{x: \|x\| = 1\}$ is complete.

b) State and prove the Hahn – Banach theorem.

Q7) a) Show that a non empty subset X of a normed linear space N is bounded if and only if $f(x)$ is a bounded set of numbers for each f in N*.

b) State and prove the closed Graph Theorem.

Q8) a) State and establish the Schwarz inequality. Deduce that the inner product in a Hilbert space is jointly continuous.

b) If T is an operator on a Hilbert space H, then show that the following conditions are equivalent to one another

i) $T^* T = I$

ii) $(Tx, Ty) = (x, y)$ for all x, y

iii) $\|T(x)\| = \|x\|$ for all x

Q9) a) If M and N are closed linear subspaces of a Hilbert space H such that $M \perp N$ then show that the linear subspace M + N is also closed.

b) If T is an operator on a Hilbert space H for which $(Tx, x) = 0$ for all x, then show that $T = 0$.

Q10) a) Prove that a Hilbert space H is separable if and only if every orthonormal set in H is countable.

b) If $P_1, P_2, \ldots, P_n$ are the projections on closed linear subspace $M_1, M_2, \ldots, M_n$ of H then $P = P_1 + P_2 + \ldots P_n$ is a projection if and only if the Pi's are pair-wise orthogonal.

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(DM22)

Total No. of Questions : 10] **[Total No. of Pages : 02**

M.Sc. DEGREE EXAMINATION, MAY – 2017

Second Year

MATHEMATICS

Measure and Integration

Time : 3 Hours **Maximum Marks: 70**

Answer any five questions

All questions carry equal marks

Q1) a) Define a countable set. If A is a countable set, then prove that the set of all finite sequences from A is also countable.

b) State and prove the Heine – Borel theorem.

Q2) a) Prove that a Borel set is measurable. Show in particular that each open set and each closed set is measurable.

b) Show that the interval (a, ∞) is measurable.

Q3) a) Prove that for each extended real number α, the set $\{x: f(x) = \alpha\}$ is measurable.

b) If m is a countably additive, translation invariant measure defined on a σ - algebra containing the set P, then prove that m [0,1) is either zero or infinite.

Q4) a) State and prove the Lebesgue convergence theorem.

b) Let f and g be integrable over E. Then prove that

i) $(f + g)$ is integrable over E and $\int_E f + g = \int_E f + \int_E g$

ii) If A and B are disjoint measurable sets in E, then $\int_{A \cup B} f = \int_A f + \int_B f$.

Q5) a) Show that if f is integrable over E, then so is $|f|$ and $\left|\int_E f\right| \leq \int_E |f|$. Does the integrability of $|f|$ imply that of f? Justify your answer.

b) State and prove the monotone convergence theorem. Show that this theorem need not hold for decreasing sequence of functions.

Q6) a) Prove that a function f is of bounded variation on $[a,b]$ if and only if f is the difference of two monotone real valued functions on $[a,b]$.

b) If f is absolutely continuous on $[a,b]$ and $f'(x) = 0$ a.e, then prove that f is constant.

Q7) a) State and prove the Holder inequality.

b) Prove that a normed linear space X is complete if and only if every absolutely summable series is summable.

Q8) a) Let $E_i \in B, \mu E_1 < \infty$ and $E_i \supset E_{i+1}$. Then show that $\mu\left(\bigcap_{i=1}^{\infty} E_i\right) = \lim_{n\to\infty} \mu E_n$.

b) Let E be a measurable set such that $0 < \nu E < \infty$. Then show that there is a positive set A contained in E with $\nu A > 0$.

Q9) a) State and prove the Hahn Decomposition Theorem.

b) Suppose that to each α in a dense set D of real numbers a set $B_\alpha \in B$ is assigned such that $B_\alpha \subset B_\beta$ for $\alpha < \beta$. Then show that there is a unique measurable extended real valued function f on X such that $f \leq \alpha$ on B_α and $f \geq \alpha$ on $X \sim B_\alpha$

Q10) a) Prove that the set function μ^* is an outer measure.

b) State and prove the Caratheodary theorem.

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Total No. of Questions : 10] **[Total No. of Pages : 02**

M.Sc. DEGREE EXAMINATION, MAY – 2017

Second Year

MATHEMATICS

Analytical Number Theory and Graph Theory

Time : 3 Hours **Maximum Marks: 70**

Answer Any five questions selecting at least two Questions from each section.

All Questions carry equal marks.

SECTION – A

Q1) a) State and prove Euler's Summation formula.

b) Prove that if d is a divisor function then for all $x \geq 1$

$$\sum_{n \leq x} d(n) = x \log x + (2c-1)x + O\left(\sqrt{x}\right)$$ where c is Euler's constant.

Q2) a) For $x > 1$ prove that

$$\sum_{n \leq x} \phi(n) = \frac{3}{\pi^2} x^2 + O(x \log x)$$

b) Prove that the set of Lattice points visible from the origin has density $\frac{6}{\pi^2}$.

Q3) Define Chebyshev's $\Psi -$ function and Chebyshev's $\theta -$ function. Obtain the relation connecting the functions Ψ and θ given by

$$0 \leq \frac{\Psi(x)}{x} - \frac{\theta(x)}{x} \leq \frac{(\log x)^2}{2\sqrt{x}\,.\log 2} \text{ for } x > 0$$

Q4) For $n \geq 2$, prove that the following inequalities hold for the functions $\pi(n)$ and log n.

$$\frac{1}{6}\frac{n}{\log n} < \pi(n) < 6\frac{n}{\log n}.$$

SECTION - B

Q5) Prove that a simple Graph with n-vertices and k-components can have atmost $\frac{(n-k)(n-k+1)}{2}$ edges.

Q6) Prove that a Graph G is an Eulerian graph if and only if all vertices of G are of even degree.

Q7) Prove that with respect to any of its spanning tree a connected graph of n-vertices and e edges has $(n-1)$ tree branches and $e - n + 1$ chords.

Q8) Prove that every circuit has an even number of edges in common with any cut-set.

Q9) Prove that any simple planar graph can be embedded in a plane such that every edge is drawn as a straight line segment.

Q10) Prove that the ring sum of two circuits in a group G is either a circuit or an edge disjoint union of circuits.

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Total No. of Questions : 10] **[Total No. of Pages : 03**

M.Sc. DEGREE EXAMINATION, MAY – 2017

Second Year

MATHEMATICS

Rings and Modules

Time : 3 Hours **Maximum Marks: 70**

Answer Any five questions.

All Questions carry equal marks.

Q1) a) Prove that a Boolean Algebra is complemented distributive lattice by defining

$$\left(a \vee b\right)' = a' \wedge b', 1 = 0',$$

conversely a complemented distributive lattice is a Boolean algebra in which these equations hold.

b) Prove that the subrings of a ring form a complete lattice under set inclusion.

Q2) a) Define a maximal ideal and prime ideal of a ring. Prove that every maximal ideal is a prime ideal. Is the converse true. Justify your answer.

b) Prove that every proper ideal of a ring is contained in a maximal ideal.

Q3) Prove that the following statements are equivalent

a) R is isomorphic to a finite direct product of rings $R_i, 1 \leq i \leq n$.

b) There exist central orthogonal idempotents $e_i \epsilon R$ such that $1 + \sum_{i=1}^{n} e_i$ and $e_i\, R \cong R$.

c) R is a finite direct sum of ideals $K_i \cong R_i$.

Q4) a) If B and C are submodules of a module A then prove that $\mathrm{B+C}/\mathrm{B}$ is isomorphic to $\mathrm{C}/\mathrm{B \cap C}$.

b) Let B be a sub module of A_R. Then prove that A_R is Noetherian if and only if B and A/B are Noetherian.

Q5) a) Let R be a ring and suppose that the ideal A of R is contained in a finite union of prime ideals $\bigcup_{i=1}^{n} P_i$. Show that A is contained in at least one of the P_i.

b) Let R be a Commutative ring. Then prove that the following conditions are equivalent.

i) R has a unique maximal ideal M.

ii) All non units of R are contained in a proper ideal in M.

iii) The non units form an ideal M.

Q6) a) If R is a commutative ring then prove that Q(R) is regular if and only if R is semi prime.

b) Prove that a Boolean algebra is isomorphic to the algebra of all subsets of a set if and only if it is complete and atomic.

Q7) a) Prove that the ring R is primitive if and only if there exists a faithful irreducible module A_R.

b) Prove that the radical is an ideal and $R/Rad\ R$ is semi primitive.

Q8) a) Prove that a ring R is completely irreducible if and only if it is isomorphic to a finite direct product of completely reducible simple rings.

b) Prove that the radical of right Artinian ring is nilpotent.

Q9) a) Prove that every free module is projective.

b) Prove that every module is isomorphic to a factor module of a projective module.

Q10) a) If M is the direct product of a family of modules $\{M_i / i \in I\}$ then prove that M is injective if and only if each M_i is injective.

b) Prove that every module is isomorphic to a sub module of an injective module.

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