II Semester M.Sc. Degree Examination, July 2017
(CBCS)
MATHEMATICS
M202T : Complex Analysis

Time : 3 Hours Max. Marks : 70

***Instructions** : 1) Answer **any five** questions.*
*2) **All** questions carry **equal** marks.*

1. a) Define harmonic functions and evaluate $\int \frac{z-3}{z^2+2z+5}dz$ where C : |z + 1 – i| = 2.

   b) Define Bilinear transformation. Find the bilinear transformation which maps the points z = 1, i, –1, into w = i, 0, – i.

   c) State and prove Morera's theorem. Also evaluate

   $\int_c \frac{e^{2z}}{z^3-1}dz$ C : |z| = 3. **(3+4+7)**

2. a) State and prove Cauchy's theorem for a rectangle.

   b) Let f(z) be analytic in a region G with zeros $a_1, a_2, \ldots, a_m$ repeated according to multiplicity. If γ is a simple closed curve in G which does not pass through any a_k, then, prove that $\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_{k=1}^{m} n(r : a_k)$.

   c) State and prove Liouville's theorem. **(6+5+3)**

3. a) Find the radius of convergence of :

   i) $\sum_{n=1}^{\infty} \frac{n!}{n^n}z^n$ ii) $\sum_{n=0}^{\infty} \frac{z^n}{n!}$.

   b) Define radius of convergence of a power series. If R is the radius of convergence of $\sum a_n z^n$, then prove the following :

   i) The power series converges for |z| < R and diverges for |z| $\geq$ R.

   ii) When R = 0, the power series diverges for all $z \neq 0 \in \mathbb{C}$ and when $R = \infty$, it converges for all $z \in \mathbb{C}$.

   c) Find the Laurent's series expansion of $f(z) = \frac{1}{z^2(z-i)}$ in

   i) 0 < | z | < 1 ii) 0 < | z – i | < 1 iii) | z – i | > 1. **(4+6+4)**

P.T.O.

4. a) State and prove Taylor's theorem.

   b) Let z = a be an isolated essential singularity of an analytic function f(z) and k = { | z – a | < r} be a neighbourhood of 'a'. For a given $\varepsilon > 0$ and any complex number ξ, prove that there exists a point z with $0 < | z - a | < r$ such that $|f(z) - \xi| < \varepsilon$.

   c) Prove that an analytic function comes arbitrarily close to any complex number in the neighbourhood of an essential singularity. **(5+6+3)**

5. a) Define residue. If f(z) has a pole of order m then prove that

$$\text{Res } f(a) = \frac{1}{(m-1)!} \lim_{z \to a} \frac{d^{m-1}}{dz^{m-1}} \left\{ (z-a)^m f(z) \right\}$$

   b) Evaluate the following :

   i) $\int_{-\infty}^{\infty} \frac{x^2}{x^4+1} dx$ ii) $\int_{-\infty}^{\infty} \frac{\sin x}{(x-1)(x^2+1)} dx$. **(4+5+5)**

6. a) Outline the argument principle, and explain why it is called by that name.

   b) Suppose f(z) and g(z) are analytic inside and on a closed curve V and $| f(z) | > g(z) | \forall z$, then show that number of zeros of f(z) + g(z) equals zeros of f(z).

   c) Show that all roots of $p(z) = z^8 + 4z^3 + 10$ lie between $1 \le | z | \le 2$. **(4+5+5)**

7. a) State and prove Phragmen Lindel of theorem.

   b) State and prove Riemann mapping theorem.

   c) Using the result of the Weier-Strass factorization theorem, construct an entire function having zero's at 1, 2, 3. **(6+5+3)**

8. a) Let f(z) be analytic in the region $| z | < \rho$ and let $z = re^{i\theta}$ be any point of this region. Then prove that

$$f(re^{i\theta}) = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{(R^2 - r^2)\, f(Re^{i\theta}) d\phi}{R^2 - 2Rr\cos(\theta + \phi) + r^2}.$$

   b) Derive the Jensen's formula with standard notations. **(8+6)**
